# A METHOD OF SOLVING THE BASIC INTEGRAL EQUATION OF STATISTICAL THEORY OF OPTIMUM SYSTEMS IN FINITE FORM

**(METOD RESHENIIA OSNOVNOGO INTEGRAL'NOGO URAVNENIIA STATISTICHESKOI TEORII OPTIMAL'NYKH SISTEM V KONECHNOI FORME)**

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We present the application of the general formula for solving a linear integral equation of the first kind, encountered in solving a number of problems of statistical theory of optimum systems [3], to the case when the kernel of the equation represents a correlation function of a random function and is related to white noise by linear differential equation. A case of an infinite observation interval is treated first and the results obtained are then applied to the case when the observation interval is finite. Finally on the basis of the author's general formula, a solution for the nonstationary case given by Laning [12] is derived in a straightforward manner. From this solution there follow previously-known solutions of specific problems by Dolph and Woodbury [11], Zadeh and Ragazzini [7,8], and Semenov. In [3] it was shown that the known results by Wiener [6] and Booton [9] for the case of infinite observation interval can be obtained from the author's general formula as special cases. Consequently, the results of this article complete the proof that all the known methods of determining optimum linear systems may be very simply obtained through the application of one general method, that of canonic representations of random functions.

**1. Introduction.** The general problem of finding an (in the statistical sense) optimum dynamic system, consists in determining with the greatest possible accuracy the value of some random function at time $s$ on the basis of observing some other random function during the time interval $s - T \leqslant t \leqslant s$. This problem is a special case of the general mathematical problem on optimum estimate of a random function $W(s)$ by transforming another realized random function $Z(t)$ observed in a region $T$ where its argument $t$ varies. Here the arguments $t$ and $s$ may be any scalar or vector variables or may even be elements of arbitrary abstract spaces. The following technical problems lead, in particular, to this problem: measurement and extrapolation of variable quantities, automatic tracking of moving objects, reception of radio signals in the presence of natural and artificial disturbances, reproduction of sound and images, design of guidance systems, machine control and industrial processes control systems, weather forecasting, etc. In solving such problems various probability criteria are used for the optimum, whose choice is determined by the specific character of any given problem. Methods of solving these problems make up the modern statistical theory of optimum systems.

Of fundamental importance in all general methods of determining the optimum operator, while using various probability criteria, is the problem of finding the linear operator $A$ which satisfies the equation of the form

$$A_t K_x(t, u) = f(s, u) \qquad (u \in T) \tag{1.1}$$

where $K_x(t, u)$ is the correlation function of some random function $X(t)$ which represents disturbance (noises, measurement errors, etc) or the sum of the disturbance and the irregularly varying part of the random function: $f(s, u)$ is the known function; index $t$ of operator $A$ indicates that the operator operates on function $K_x$ when the latter is taken to be a function of $t$ with $u$ held fixed. Equation (1.1) must be satisfied for all values of $u$ of observation region $T$.

Equation (1.1) was originated as an equation determining the optimum linear operator for the criterion of minimum mean-square error [1]. Andreev has shown that the same equation also governs the problem of determining of the optimum linear operator by the more general criterion of extremum of the given function of mathematical expectancy and error dispersion [5] (see also [1]). A special case of equation (1.1) was obtained [12] while determining the optimum operator by the criterion of minimum mathematical expectancy of the given error function for the normally distributed disturbance, and also when solving various special problems in the theory of optimum systems in a number of other papers. A general method of determining the optimum operator for the case of normally distributed disturbances and using an arbitrary criterion of the Bayes-type has recently been developed. This method also is based on solving equations of form (1.1) [4].

Equation (1.1) has been solved for various special cases in a number of papers [6-12]. In [2] a general solution of equation (1.1) is given in the form of an infinite series, and is obtained by the method of canonic expansions of random functions (see also [1]). This solution makes it possible in all cases (for any scalar or vector functions $X(t)$ and $f(s, u)$ and for arbitrary scalar or vector variables $t$, $s$, $u$), to find an approximate solution of equation (1.1) suitable for numerical computations. In particular, this solution makes it possible to determine optimum one-dimensional (one input and one output) and multidimensional (several inputs and outputs) automatic linear systems designed to reproduce signals in the presence of disturbances. In [ 3 ] a general solution of equation (1.1) is given in closed form, derived by the method of integral canonic representations of random functions. The practical application of this solution is limited to those cases when it is possible to find integral canonic representations of random function $X$. From this general solution (3), in particular, there was obtained the formula for the weighting function (kernel) of the linear integral operator $A$ satisfying equation (1.1) when both $t$ and $s$ are continuously varying scalars (in special case time moments) and the observation region $T$ is an infinite interval $-\infty < t \leqslant s$. In this case equation (1.1) is a linear integral equation of the first kind

$$\int_{-\infty}^{s} K_x(t, u)\, g(s, t)\, dt = f(s, u) \qquad (-\infty < u \leqslant s) \tag{1.2}$$

where $g(s, t)$ is the weighting function of the desired linear integral operator $A$. The solution of equation (1.2) obtained in [ 3 ] is of the form

$$g(s, t) = \int_{-\infty}^{s} \frac{w^-(\lambda, t)\, d\lambda}{G(\lambda)} \int_{-\infty}^{s} f(s, u)\, w^-(\lambda, u)\, du \tag{1.3}$$

where $w^-(t, \tau)$ is the weighting function of the linear system which transforms random function $X(t)$ into white noise $V(t)$, and $G(t)$ is the dispersion density of white noise $V(t)$. A linear system with weighting function $w^-(t, \tau)$ is the inverse of a linear system with weighting function $w(t, \tau)$ forming a given function $X(t)$ from white noise $V(t)$.

In [ 3 ] it has been shown that from the general formula (1.3) there follow as special cases previously-known formulas by Wiener (6) and Booton (9). In this article we will show that from formula (1.3) there follow all the known closed-form solutions of equation (1.1) obtained in solving problems of determining optimum one-dimensional linear systems for the finite observation interval $s - T \leqslant t \leqslant s$. This concludes the development of the general theory of solving equations of the form (1.1), based on a unique mathematical method - the method of canonic representations of random functions. Thus the results of this article, together with the results of papers [ 1-4 ], will permit us to state that the method of canonic representations of random functions is the foundation of the modern statistical theory of omptimum systems.

**2. Case when random function $X$ is related to white noise through a linear differential equation and the observation interval is infinite.** Let us consider a special case when random function $X(t)$ is related to white noise $V(t)$ through a linear differential equation

$$F_t X(t) = H_t V(t) \qquad \left(F_t = \sum_{k=0}^{n} a_k D^k,\ H_t = \sum_{k=0}^{m} b_k D^k,\ D = \frac{d}{dt}\right) \tag{2.1}$$

whose coefficients, in general, may be arbitrary functions of time $t$, possessing all the derivatives necessary for further calculations and moreover, $n > m$ (the case when $n \leqslant m$ has no practical value, since the dispersion of random function $X$ is infinite). Moreover, let us first consider a case when the observation interval is infinite, $-\infty < t \leqslant s$. Evidently, we can, without loss of generality, take dispersion density $G(t)$ of white noise $V$ as identically equal to unity, since the general case can always be put into this form through the substitution $V(t) = \sqrt{G(t)}\, V_1(t)$. Then (1.3) will assume the form

$$g(s, t) = \int_{-\infty}^{s} w^-(\lambda, t)\, d\lambda \int_{-\infty}^{s} f(s, u)\, w^-(\lambda, u)\, du \tag{2.2}$$

In the case considered, weighting function $w^-$ is defined by the differential equation

$$H_t w^-(t, \tau) = F_t \delta(t - \tau) \tag{2.3}$$

Let us introduce a weighting function $p(t, \tau)$, corresponding to operator $H$. It satisfies the following differential equation

$$H_t p(t, \tau) = \delta(t - \tau) \tag{2.4}$$

Then we will obtain*

$$w^-(\lambda, t) = F_t^* p(\lambda, t) \tag{2.5}$$

and (2.2) will assume the form

$$g(s, t) = F_t^* \int_{-\infty}^{s} p(\lambda, t)\, \xi(s, \lambda)\, d\lambda, \qquad \xi(s, \lambda) = \int_{-\infty}^{s} w^-(\lambda, u)\, f(s, u)\, du \tag{2.6}$$

This formula may be represented in the form

$$g(s, t) = F_t^* \eta(s, t), \qquad \eta(s, t) = \int_{-\infty}^{s} p(\lambda, t)\, \xi(s, \lambda)\, d\lambda \tag{2.7}$$

Thus formula (2.2) is equivalent to the three formulas (2.6), (2.7). From these formulas in the given order functions $\xi$, $\eta$ and g can be computed.

Since $n > m$, the weighting function $w$ defined by the differential

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* The asterisk denotes the corresponding conjugate differential operators throughout.

equation (2.3) must contain $\delta$-functions and all its derivatives up to and including the order $n - m$. Consequently, the function $\xi$, defined by (2.6), will contain a linear combination of function $f$ and all its derivatives up to the order $n - m$, inclusive. The remainder of the right-hand side of formula (2.6) becomes zero when $\lambda = -\infty$. Thus formula (2.6) defines function $\xi$ as a solution of the differential equation

$$H_t \xi(s, t) = F_t f(s, t) \tag{2.8}$$

and this solution is a sum of a linear combination of function $f$ and its first $n - m$ derivatives, and some function which becomes zero for $t = -\infty$.

The formula (2.7) defines the function $\eta$ as the solution of the differential equation

$$H_t^* \eta(s, t) = \xi(s, t) \tag{2.9}$$

and this solution becomes zero for $t = s$ together with its first $m - 1$ derivatives, i.e. it satisfies the end conditions

$$\eta(s, s) = \eta_t'(s, s) = \ldots = \eta_t^{(m-1)}(s, s) = 0 \tag{2.10}$$

Consequently, to determine the weighting function $g$ satisfying equation (1.2), one must find the solution of equation (2.8) (which is a sum of a linear combination of function $f$ and its first $n - m$ derivatives and a function which becomes zero for $t = -\infty$ ) and solve equation (2.9) subject to the end-conditions (2.10). Weighting function $g$ can then be determined from (2.7) by means of differentiation, multiplication, and addition.

The derivative of the $m$-th order of the function $\eta$, in general, will have a discontinuity of the first kind at $t = s$. Therefore, since $n > m$, the function $g$ defined by (2.7) in general will contain a linear combination of $\delta$-function and its derivatives of the order $n - m - 1$ inclusive. This linear combination of $\delta$-functions can be separated from the rest of function $g$ as follows

$$g(s, t) = g_1(s, t) + \sum_{r=0}^{n-m-1} B_r \delta^{(r)}(t - s) \tag{2.11}$$

where $g_1$ is a function which has no $\delta$-functions. Coefficients $B_r$ in (2.11) are expressed through discontinuities of the derivatives of function $\eta$ of the order higher than $m - 1$ and values of the $F$-operator coefficients and their derivatives at $t = s$. Thus they can be expressed through the following formula [ 12 ]

$$B_r = \sum_{h=m}^{n-r-1} \sum_{l=m}^{h} (-1)^{h+r+1} C_h^l a_{h+r+1}^{(h-l)}(s) \Delta_1 \eta^{(l)} \quad (r = 0, 1, \ldots, n-m-1) \tag{2.12}$$

where discontinuities of the derivatives of function $\eta$ are defined by

$$\Delta_1 \eta^{(l)} = -\eta_t^{(l)}(s, s) \quad (l = m, \ldots, n-1) \tag{2.13}$$

In a special case when operator $H$ is unity, $H = 1$, formulas (2.7), (2.8), (2.9), and (2.11) give the solution of equation (1.2) explicitly

$$g(s, t) = F_t^* F_t f(s, t) + \sum_{r=0}^{n-1} B_r \delta^{(r)}(t - s) \tag{2.14}$$

**3. Case when random function $X(t)$ is related to white noise through a linear differential equation and the observation interval is finite.** When the observation interval is finite, $s - T \leqslant t \leqslant s$, and there is a linear integral operator $A$, equation (1.1) is a linear integral equation of the first kind

$$\int_{s-T}^{s} K_x(t, u)\, g(s, t)\, dt = f(s, u) \quad (s - T \leqslant u \leqslant s) \tag{3.1}$$

Formula (1.3) and the whole method described in the preceding section are not directly applicable to equation (3.1). However, equation (3.1) can be transformed into (1.2) by extending function $f$ into the region $u < s - T$ in such a fashion that function $g$ (defined by (1.2) ) becomes zero for all $t < s - T$. Equations (1.2) and (3.1) will then be equivalent in the interval $s - T \leqslant u \leqslant s$ and their solutions will be identical. On the basis of (2.7) the condition

$$g(s, t) = 0 \quad \text{for } t < s - T \tag{3.2}$$

is identical to the condition

$$F_t^* \eta(s, t) = 0 \quad \text{for } t < s - T \tag{3.3}$$

Consequently, the problem in this case is to solve (3.3), compute function $\xi$ for $t < s - T$ using (2.9), and to determine function $f$ for $t < s - T$ by solving equation (2.8). Function $g$ will then be determined by the method of the preceding section.

Let $\eta_1, \ldots, \eta_n$ be some linearly independent solutions of equation (3.3). Its general solution will then be expressed as follows

$$\eta(s, t) = \sum_{r=1}^{n} c_r \eta_r(t) \quad (t < s - T) \tag{3.4}$$

Substituting this into (2.9) we obtain

$$\xi(s, t) = \sum_{r=1}^{n} c_r \xi_r(t) \quad (t < s - T) \tag{3.5}$$

where

$$\xi_r(t) = H_t^* \eta_r(t) \quad (r = 1, \ldots, n;\ t < s - T) \tag{3.6}$$

Substituting (3.5) into (2.6) with $\lambda = \tau$; multiplying it by weighting function $w(t, \tau)$, which serves to transform white noise into random function $X(t)$; integrating the result with respect to $\tau$ and between the limits $-\infty$ and $t$; and taking into consideration the known relation between weighting functions of mutually inverse systems we obtain

$$f(s, t) = \sum_{r=1}^{n} c_r f_r(t), \quad f_r(t) = \int_{-\infty}^{t} w(t, \tau)\, \xi_r(\tau)\, d\tau \quad \begin{pmatrix} t < s - T \\ r = 1, \ldots, n \end{pmatrix} \tag{3.7}$$

Equation (3.7) defines function $f$ for $t < s - T$, with an accuracy within $n$ arbitrary constants $c_r$. To determine conditions from which these constants can be found, let us note that function $g$ (defined by (2.7) ) cannot have derivatives of the $\delta$-function of an order higher than $n - m - 1$. Otherwise the transformation result of the random function under observation (transformed by means of the integral operator $A$ which has weighting function $g$) will have a component in the form of white noise and its dispersion will be infinite. To satisfy this condition it is necessary and sufficient for function $\eta$ to be continuous at $t = s - T$, together with its first $m - 1$ derivatives. For this, in turn, as is seen from equation (2.9), it is necessary and sufficient for function $\xi$ to be continuous and have a discontinuity of the first kind at $t = s - T$. But function $\xi$ is defined by (2.6) and, as was shown in the preceding section, it contains a linear combination of function $f$ and its first $n - m$ derivatives. Therefore, function $\xi$ will satisfy this condition only if function $f$ and its derivatives of the order $n - m - 1$ inclusive are continuous at $t = s - T$. This condition will give $n - m$ equations relating values of $c_r$. The other $m$ equations will be obtained from the condition of coincidence of function $\eta$ (defined by (2.7) ) and the solution (3.4) of equation (3.3) for $t < s - T$. We will obtain these equations later, leaving values of $c_r$ undetermined at this point. Dividing the interval of integration of equation (2.6) into two parts, $-\infty < u < s - T$ and $s - T \leqslant u \leqslant s$, and utilizing expression (3.7) of function $f$ for $t < s - T$ we obtain

$$\xi(s, t) = \sum_{r=1}^{n} c_r \xi_r(t) + \int_{s-T}^{t} w^{-}(t, u)\, f(s, u)\, du \qquad (s - T \leqslant t \leqslant s) \tag{3.8}$$

where

$$\xi_r(t) = \int_{-\infty}^{s-T} w^{-}(t, u)\, f_r(u)\, du \qquad (r = 1, \ldots, n;\ s - T \leqslant t \leqslant s) \tag{3.9}$$

To determine function $\eta$ for $s - T \leqslant t \leqslant s$ from (2.7), it is first necessary to find weighting function $p$. For this, let us treat it as a solution of the equation

$$H_t^{*} p(\lambda, t) = \delta(\lambda - t) \tag{3.10}$$

which is conjugate to (2.4). Denoting through $y_1, \dots, y_m$ the solutions (linearly independent) of the equation

$$H_t^* y = 0 \tag{3.11}$$

which satisfy the initial conditions

$$y_k^{(h-1)}(s-T) = \delta_{kh} \qquad (k, h = 1, \dots, m) \tag{3.12}$$

function $p$ can be represented by

$$p(\lambda, t) = \begin{cases} \sum_{l=1}^{m} p_l(\lambda)\, y_l(t) & \text{for } \lambda > t \\ 0 & \text{for } \lambda < t \end{cases} \tag{3.13}$$

$$\left(p_l(\lambda) = \frac{(-1)^{l+1}}{a_m(\lambda)} \frac{W[y_1, \dots, y_{l-1}, y_{l+1}, \dots, y_m]}{W[y_1, \dots, y_m]}\right)$$

Here $W[\phi_1, \dots, \phi_k]$ denotes the Wronskian of the functions $\phi_1(\lambda), \dots, \phi_k(\lambda)$.

Substituting the expression of weighting function $p$ (3.13) and the expression of function $\xi$ (3.8) into (2.7) we obtain

$$\eta(s, t) = \sum_{l=1}^{m} y_l(t)\left[\sum_{r=1}^{n} c_r z_{lr}(t) + u_l(t)\right] \quad (s-T \leqslant t \leqslant s) \tag{3.14}$$

where

$$z_{lr}(t) = \int_t^s p_l(\lambda)\, \xi_r(\lambda)\, d\lambda \qquad \begin{pmatrix} l = 1, \dots, m \\ r = 1, \dots, n \end{pmatrix} \tag{3.15}$$

$$u_l(t) = \int_t^s p_l(\lambda)\, d\lambda \int_{s-T}^{s} w^-(\lambda, u)\, f(s, u)\, du \qquad (l = 1, \dots, m) \tag{3.16}$$

Dividing the integration interval into two parts and expressing function $\xi$ for $t < s - T$ through (3.5), for $t < s - T$ we have

$$\eta(s, t) = \sum_{l=1}^{m} y_l(t)\left[\sum_{r=1}^{n} c_r z_{lr}(s-T) + u_l(s-T)\right] + \sum_{r=1}^{n} c_r \psi_r(t \tag{3.17}$$

where

$$\psi_r(t) = \int_t^{s-T} p(\lambda, t)\, \xi_r(\lambda)\, d\lambda \qquad (r = 1, \dots, n) \tag{3.18}$$

It is obvious that $\psi_r(t)$ is the solution of equation

$$H_t^* \psi_r(t) = \xi_r(t) = H_t^* \eta_r(t) \qquad (r = 1, \dots, n) \tag{3.19}$$

and it becomes zero, together with its first $m-1$ derivatives, for $t = s-T$. Consequently,

$$\psi_r(t) = \eta_r(t) - \sum_{l=1}^{m} \eta_r^{(l-1)}(s-T)\, y_l(t) \qquad (r = 1, \ldots, n) \tag{3.20}$$

Substituting this expression into (3.17) we obtain

$$\eta(s, t) = \sum_{l=1}^{m} y_l(t) \left\{ \sum_{r=1}^{n} c_r [z_{lr}(s-T) - \eta_r^{(l-1)}(s-T)] + \right.$$
$$\left. + u_l(s-T) \right\} + \sum_{r=1}^{n} c_r \eta_r(t) \tag{3.21}$$

Comparing this with (3.4) we see that function $\eta$ as defined for $t < s-T$ by (3.21) coincides with function $\eta$ as previously defined as a solution of equation (3.3) only when the following conditions are satisfied

$$\sum_{r=1}^{n} c_r [z_{lr}(s-T) - \eta_r^{(l-1)}(s-T)] = -u_l(s-T) \qquad (l = 1, \ldots, m) \tag{3.22}$$

From the continuity condition of function $f$ and its first $n-m-1$ derivatives at $t = s-T$, the remaining equations for determining $c_r$ can be written as

$$\sum_{r=1}^{n} c_r f_r^{(k)}(s-T) = f_t^{(k)}(s, s-T) \qquad (k = 0, 1, \ldots, n-m-1) \tag{3.23}$$

Once the values of $c_r$ have been established through solving a system of linear algebraic equations (3.22) and (3.23), formulas (3.14) and (2.7) will completely define the solution of equation (3.1).

In the general case the above method gives function $\eta$ whose $m$-th derivative possesses discontinuities of the first kind at $t = s-T$ and $t = s$. Therefore, weighting function $g$ (defined by (2.7) ) will in general contain a linear combination of the $\delta$-function and its derivatives of the $n-m-1$ order, inclusive, corresponding to points $t = s-T$ and $t = s$. Separating this linear combination of $\delta$-functions from the rest of the $g$-function, we obtain

$$g(s, t) = g_1(s, t) + \sum_{r=0}^{n-m-1} [A_r \delta^{(r)}(t-s+T) + B_r \delta^{(r)}(t-s)] \tag{3.24}$$

where $g_1$ is a function having no $\delta$-functions, the coefficients $B_r$ are determined from (2.12) and (2.13), and the coefficients $A_r$ are determined from the similar formulas

$$A_r = \sum_{h=m}^{n-r-1} \sum_{l=m}^{h} (-1)^{h+r+1} C_h^l a_{h+r+1}^{(h-l)} (s-T) \Delta_0 \eta^{(l)} \tag{3.25}$$

$$(r = 0, 1, \ldots, n-m-1)$$

$$\Delta_0 \eta^{(l)} = \eta_l^{(l)}(s, s-T+0) - \eta_l^{(l)}(s, s-T-0) \quad (l = m, \ldots, n=1) \tag{3.26}$$

This method of solving integral equation (3.1) differs somewhat from the Laning method, although it naturally leads to the same results. The derivation of the solution of equation (3.1) on the basis of the general formula (2.2) obtained by the method of integral canonic representations of random functions is considerably simpler than Laning's formal derivation. This method differs from Laning's, for example, in that it requires solution of $n$ linear algebraic equations with $n$ unknown constants, while the Laning method necessitates solution of a system of $n + m$ linear algebraic equations with $n + m$ unknown constants. These differences are not fundamental, of course. They stem from the fact that in the Laning method the unknown functions $\xi$ and $\eta$ for $t > s - T$ are determined by solving equations (2.8) and (2.9), while this method is based on the application of formulas (2.6) and (2.7), leading directly to the desired particular solutions of equations (2.8) and (2.9). This reduces the number of undetermined values and of equations required to determine them.

In a special case when operator $H$ is unity, $H = 1$, this method gives the known solution of equation (3.1) obtained by Dolph and Woodbury [11]

$$g(s, t) = F_t^* F_t f(s, t) + \sum_{r=0}^{n-1} [A_r \delta^{(r)}(t-s+T) + B_r \delta^{(r)}(t-s)] \tag{3.27}$$

In a special case, when coefficients $a_k$ and $b_k$ determining the linear differential operators $F$ and $H$ in (2.1) are constant, the random function $X$ defined by differential equation (2.1) is a stationary random function with fractional rational spectral density. Also, all differential equations (2.8), (2.9), (3.3), and (3.11) will be linear differential equations with constant coefficients, and consequently can be solved by known standard methods. If, moreover, function $f(s, t)$ is a polynomial in $t$, then this method gives the known solution of the problem by Zadeh and Ragazzini (7,8) and by V.M. Semenov.

This method, as well as formulas (1.3) and (2.2), is easily generalized to include the case when random process $X(t)$ defined by equation (2.1) is not infinite but starts at some finite instant $t_0 < s - T$. To obtain the solution of equation (3.1) for this case, $-\infty$ should be replaced by $t_0$ in all the formulas in this article. Incidentally, this gives an interesting generalization of the problem, by Zadeh and Ragazzini, for nonstationary random functions related to white noise through linear differential equations with constant coefficients.

This method may be generalized to include the case of a vector random function $X$, whose components are expressed through noncorrelated white noises by a system of linear differential equations. To obtain this generalization, it is sufficient to utilize the general formula for weighting functions, satisfying the system of integral equations of the first kind into which equation (1.1) transforms in this case (such a formula was obtained in [3] by the method of integral canonic representations of random functions) and to apply to this formula the considerations by which this method was derived from (2.2).

**Example 1.** Find solution of equation (3.1) when $T = s$ and operators $F$ and $H$ in equation (2.1) are expressed as

$$F = a_1(t)D + a_0(t), \qquad H = 1 \tag{3.28}$$

In this case the solution of equation (3.1) is determined from (3.27) for $n = 1$ and $T = s$. To find coefficients $A_0$ and $B_0$ of the $\delta$-functions it is necessary to determine the function and its discontinuities at the points $t = 0$ and $t = s$. For this we shall need the weighting function $w(t, \tau)$, corresponding to (2.1). It is easy to see that in this case it is determined by

$$w(t, \tau) = \frac{q_1(t)}{a_1(t)\, q_1(\tau)}, \qquad q_1(t) = \exp\left(-\int_0^t \frac{a_0(\tau)}{a_1(\tau)}\, d\tau\right) \qquad (t > \tau) \tag{3.29}$$

Equation (3.3) for this case has the form

$$-\frac{d}{dt}[a_1(t)\eta] + a_0(t)\eta = 0 \tag{3.30}$$

This equation is of the first order, and consequently in (3.4), (3.5), and (3.7) $n = 1$. Solving (3.30) and applying (3.6) we find

$$\xi_1(t) = \eta_1(t) = \frac{1}{a_1(t)\, q_1(t)} \tag{3.31}$$

Substituting (3.29) and (3.31) into (3.7), we obtain

$$f_1(t) = \int_{-\infty}^{t} \frac{q_1(t)}{a_1^2(\tau)\, q_1^2(\tau)}\, d\tau = q_1(t) \int_{-\infty}^{t} \frac{d\tau}{a_1^2(\tau)\, q_1^2(\tau)} = q_2(t) \tag{3.32}$$

Consequently, formulas (3.4) and (3.7) defining functions $\eta$ and $f$ for $t < s - T = 0$ will in this case have the form

$$\eta(s, t) = \frac{c_1}{a_1(t)\, q_1(t)}, \qquad f(s, t) = c_1 q_2(t) \qquad (t < 0) \tag{3.33}$$

To determine the unknown constant $c_1$ we have one equation (3.23) which constitutes the continuity condition of the function $f$ at $t = 0$. From this equation we find

$$c_1 = \frac{f(s, 0)}{q_2(0)} \tag{3.34}$$

For $t > 0$ the function $\eta$ is defined by equations (2.8) and (2.9) which give

$$\eta(s, t) = \xi(s, t) = F_t f(s, t) = a_1(t) f'_t(s, t) + a_0(t) f(s, t) \qquad (t > 0) \tag{3.35}$$

Using (3.26), (3.33), (3.34), and (3.35) and taking into consideration (3.29) and (3.32) we obtain the following expression for discontinuity of function $\eta$ at $t = 0$

$$\begin{aligned}\Delta_0 \eta = a_1(0) f'_t(s, 0) + a_0(0) f(s, 0) - \frac{f(s, 0)}{a_1(0) q_1(0) q_2(0)} = \\ = a_1(0)\left[f'_t(s, 0) - \frac{q'_2(0)}{q_2(0)} f(s, 0)\right]\end{aligned} \tag{3.36}$$

Similarly, using (2.13), (3.35), and (3.29) we find the discontinuity of function $\eta$ at $t = s$

$$\Delta_1 \eta = -a_1(s) f'_t(s, s) - a_0(s) f(s, s) = -a_1(s)\left[f'_t(s, s) - \frac{q'_1(s)}{q_1(s)} f(s, s)\right] \tag{3.37}$$

Once the discontinuities of function $\eta$ are found, the coefficients of $\delta$-functions are determined from (3.25) and (2.14)

$$\begin{aligned}A_0 &= -a_1(0)\,\Delta_0 \eta = -a_1^2(0)\left[f'_t(s, 0) - \frac{q'_2(0)}{q_2(0)} f(s, 0)\right] \\ B_0 &= -a_1(s)\,\Delta_1 \eta = a_1^2(s)\left[f'_t(s, s) - \frac{q'_1(s)}{q_1(s)} f(s, s)\right]\end{aligned} \tag{3.38}$$

Substituting expressions for $A_0$ and $B_0$ into (3.27) and taking (3.29) into consideration, we find the desired solution of equation (3.1)

$$\begin{aligned}g(s, t) = -a_1^2(t) f''_t(s, t) - 2a_1(t) a'_1(t) f'_t(s, t) + [a_0^2(t) - a'_0(t) a_1(t) - a_0(t) a'_1(t)] f(s, t) - \\ - a_1^2(0)\left[f'_t(s, 0) - \frac{q'_2(0)}{q_2(0)} f(s, 0)\right]\delta(t) + a_1^2(s)\left[f'_t(s, s) - \frac{q'_1(s)}{q_1(s)} f(s, s)\right]\delta(t - s)\end{aligned} \tag{3.39}$$

This formula was first obtained by a different method by Dolph and Woodbury [ 11 ].

**Example 2.** Find solution of equation (3.1) when $T = s$ and operators $F$ and $H$ of (2.1) and function $f$ are as follows

$$F = D^2 + 2aD + b^2, \qquad H = ke^{\mu t}(D + b), \qquad f(s, t) = \lambda_1 + \lambda_2 t \tag{3.40}$$

where $a$, $b$, $k$, $\mu$, are constants and $b > a > 0$.

This problem is encountered in optimizing a linear system designed to reproduce linear time functions with random coefficients and using minimum mean-square error as a criterion [ 1 ].

In this case equation (3.3) has the form

$$\eta'' - 2a\eta' + b^2\eta = 0 \tag{3.41}$$

Two of its linearly independent solutions are

$$\eta_1(t) = e^{(a+i\omega_0)t}, \qquad \eta_2(t) = e^{(a-i\omega_0)t} \qquad (\omega_0 = \sqrt{b^2 - a^2}) \tag{3.42}$$

Substituting (3.42) into (3.6), we find

$$\xi_1(t) = k(b - a_1 - i\omega_0)\, e^{(a_1+i\omega_0)t}, \quad \xi_2(t) = k(b - a_1 + i\omega_0)\, e^{(a_1-i\omega_0)t} \tag{3.43}$$

where for the sake of brevity,

$$a_1 = a + \mu$$

To determine $f_1$ and $f_2$ from (3.7), we first find weighting function which in this case is determined by

$$w_t''(t, \tau) + 2aw_t'(t, \tau) + b^2w(t, \tau) = ke^{\mu t}[\delta'(t-\tau) + b\delta(t-\tau)] \tag{3.44}$$

Solving this equation, we obtain (for $t > \tau$)

$$w(t, \tau) = \frac{k}{2i\omega_0}[(b - a_1 + i\omega_0)\, e^{-(a-i\omega_0)t+(a_1-i\omega_0)\tau} - \\ - (b - a_1 - i\omega_0)\, e^{-(a+i\omega_0)t+(a_1+i\omega_0)\tau}] \tag{3.45}$$

Substituting (3.43) and (3.45) into (3.7), we find functions $f_1$ and $f_2$

$$f_1(t) = \overline{f_2(t)} = -\frac{k^2}{2a_1}\left[i\omega_0 + \frac{\mu(2a+\mu)}{2(a_1+i\omega_0)}\right] \bar{e}^{(a_2+i\omega_0)t} \tag{3.46}$$

where

$$a_2 = a_1 + \mu = a + 2\mu$$

To determine functions $\xi_1$ and $\xi_2$ for $t > 0$, we must first find weighting function $w^-$. Its equation (2.3) for this case has the form

$$ke^{\mu t}\left[\frac{\partial w^-(t,\tau)}{\partial t} + bw^-(t,\tau)\right] = \delta''(t-\tau) + 2a\delta'(t-\tau) + b^2\delta(t-\tau) \tag{3.47}$$

Solving it, we obtain

$$w^-(t,\tau) = \frac{e^{-\mu t}}{k}[\delta'(t-\tau) + (2a - b + \mu)\,\delta(t-\tau)] + \\ + \frac{e^{-\mu\tau}}{k}[2b(b-a) - \mu(2b - 2a - \mu)]\, e^{-b(t-\tau)}\, 1(t-\tau) \tag{3.48}$$

Substituting (3.46) and (3.48) into (3.9), we find expressions for functions $\xi_1$ and $\xi_2$ for $t > 0$

$$\xi_1(t) = \overline{\xi_2(t)} = -k\nu e^{-bt}, \qquad \nu = \frac{2b(b-a) - \mu(2b-2a-\mu)}{2a_1(a_1+b+i\omega_0)}\left[i\omega_0 + \frac{\mu(2a+\mu)}{2(a_1+i\omega_0)}\right] \tag{3.49}$$

For further computations we must find solutions of equations (3.11) satisfying conditions (3.12). Equation (3.11) for this case has the form

$$y' - (b-\mu)y = 0 \tag{3.50}$$

The solution of this equation satisfying the initial condition (3.12) has the form

$$y_1(t) = e^{(b-\mu)t} \tag{3.51}$$

Weighting function $p$, defined by (2.4) will for this case be

$$p(t,\tau) = \frac{1}{k}\, e^{-bt}\, e^{(b-\mu)\tau} \tag{3.52}$$

Comparing this with (3.13), we find

$$p_1(\tau) = \frac{1}{k}\, e^{-b\tau} \tag{3.53}$$

Substituting (3.49) and (3.53) into (3.15), we find functions $z_{11}$ and $z_{12}$

$$z_{11}(t) = \overline{z_{12}(t)} = -\frac{\nu}{2b}\left(e^{-2bt} - e^{-2bs}\right) \tag{3.54}$$

Substituting (3.48), (3.53) and (3.40) into (3.16) we find function $u_1$

$$u_1(t) = [\beta_1\lambda_2 + \beta_2(\lambda_1 + \lambda_2 t)]\left[e^{-(b+\mu)t} - e^{-(b+\mu)s}\right] - \beta_3[(b-\mu)\lambda_1 - \lambda_2]\left(e^{-2bt} - e^{-2bs}\right) \tag{3.55}$$

where

$$\beta_1 = 2\frac{a(b^2-\mu^2) - b^2\mu}{k^2(b^2-\mu^2)^2}, \quad \beta_2 = \frac{b^2}{k^2(b^2-\mu^2)}, \quad \beta_3 = \frac{2b(b-a) - \mu(2b-2a-\mu)}{2k^2b(b-\mu)^2} \tag{3.56}$$

The formula (3.14), which determined function $\eta$ for $t > 0$ for this case, has the form

$$\eta(s,t) = e^{(b-\mu)t}[c_1 z_{11}(t) + c_2 z_{12}(t) + u_1(t)] \tag{3.57}$$

Equations (3.22) and (3.23), defining $c_1$ and $c_2$ have the form

$$c_1[z_{11}(0) - 1] + c_2[z_{12}(0) - 1] = -u_1(0), \quad c_1 f_1(0) + c_2 f_2(0) = \lambda_1 \tag{3.58}$$

Once these equations are solved and the values of $c_1$ and $c_2$ are substituted into (3.57), function $\eta$ will be completely defined. All that remains, to determine the desired weighting function $g(s, t)$, is to find the coefficients of $\delta$-functions in (3.24). Using (2.12), (2.13), (3.25), (3.26) and (3.42), we find

$$A_0 = \Delta_0\eta' = \eta_t'(s, 0) - c_1(a + i\omega_0) - c_2(a - i\omega_0), \qquad B_0 = \Delta_1\eta' = -\eta_t'(s,s) \tag{3.59}$$

On the basis of (2.7), (3.24) and (3.59), the weighting function of the optimum linear system will be expressed through the formula

$$g(s, t) = \eta_t''(s,t) - 2a\eta_t'(s, t) + b^2\eta(s, t) + [\eta_t'(s, 0) - c_1(a + i\omega_0) - c_2(a - i\omega_0)]\,\delta(t) - \\ - \eta_t'(s, s)\delta(t - s) \tag{3.60}$$

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